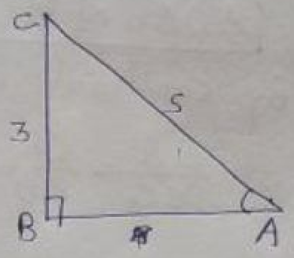
Chapter 18.

Trigonometric Identities

1.) If A is an acute angle and $\sin A = \frac{3}{5}$, find all other trigonometric ratios of angle A.



→ Given that, $\sin A = \frac{3}{5}$

In $\triangle ABC$, $\angle B = 90^\circ$

$AC = 5$ cm and $BC = 3$ cm

Then, By Pythagoras Theorem,

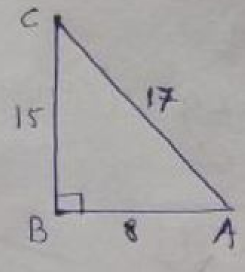
$$AC^2 = AB^2 + BC^2$$

$$AB^2 = AC^2 - BC^2$$

$$\Rightarrow AB = \sqrt{AC^2 - BC^2} = \sqrt{25-9} = \sqrt{16} = 4 \text{ cm}$$

Then, $\left\{ \begin{array}{ll} \cos A = \frac{AB}{AC} = \frac{4}{5} & \sec A = \frac{1}{\cos A} = \frac{5}{4} \\ \tan A = \frac{BC}{AB} = \frac{3}{4} & \operatorname{cosec} A = \frac{1}{\sin A} = \frac{5}{3} \end{array} \right\}$

2.) If A is an acute angle and $\sec A = \frac{17}{8}$, find all other trigonometric ratios of angle A.



→ Given that, A is an acute angle

$\sec A = \frac{17}{8}$

In $\triangle ABC$, $AC = 17$ cm, $AB = 8$ cm

By Pythagoras Theorem,

$$AC^2 = AB^2 + BC^2$$

$$BC^2 = AC^2 - AB^2 \Rightarrow BC = \sqrt{AC^2 - AB^2} = \sqrt{289-64}$$

$$BC = \sqrt{225} = 15 \text{ cm}$$

Then, $\left\{\begin{array}{ll} \sin A = \frac{BC}{AC} = \frac{15}{17} & \cot A = 1/\tan A = \frac{8}{15} \\ \cos A = \frac{1}{\sec A} = \frac{8}{17} & \operatorname{cosec} A = \frac{1}{\sin A} = \frac{17}{15} \\ \tan A = \frac{BC}{AB} = \frac{15}{8} & \end{array}\right\}$

3.) If $12 \operatorname{cosec}\theta = 13$, find the value of $\frac{(2\sin\theta - 3\cos\theta)}{(4\sin\theta - 9\cos\theta)}$

→ Given that, $12\operatorname{cosec}\theta = 13$

$\operatorname{cosec}\theta = \frac{13}{12}$

In ΔABC, $\angle A = \theta$

Then, $\operatorname{cosec}\theta = \frac{AC}{BC} = \frac{13}{12}$ ⇒ $AC = 13$ and $BC = 12$

By Paythagoras Theorem,

$AB = \sqrt{AC^2 - BC^2} = \sqrt{13^2 - 12^2} = \sqrt{169 - 144} = \sqrt{25} = 5$

$\boxed{AB = 5cm}$

Then, $\sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{12}{13}$ and $\cos\theta = \frac{AB}{AC} = \frac{5}{13}$

$L.H.S. = \frac{(2\sin\theta - 3\cos\theta)}{(4\sin\theta - 9\cos\theta)}$

$= \frac{2 \times 12/13 - 3 \times 5/13}{4 \times 12/13 - 9 \times 5/13} = \frac{(24-15)}{(48-45)} = \frac{9}{3} = 3$

Thus, $\boxed{\frac{(2\sin\theta - 3\cos\theta)}{(4\sin\theta - 9\cos\theta)} = 3}$

4.) i) $\cos^2(26°) + \cos(64°) \cdot \sin(26°) + (\tan 36°/\cot 54°)$

ii) $(\sec 17°/\operatorname{cosec} 73°) + (\tan 68°/\cot 22°) + \cos^2 44° + \cos^2 46°$.

→ i) Given that,

$= \cos^2(26°) + \cos(64°) \cdot \sin(26°) + (\tan 36°/\cot 54°)$

$= \cos^2(26°) + \cos(90-16) \cdot \sin 26° + [\tan 36°/\cot(90° - 54°)]$

$= [\cos^2(26°) + \sin^2(26°)] + [\tan 36°/\tan 36°]$

$= 1 + 1 = 2$

ii) $(\sec 17^\circ / \operatorname{cosec} 73^\circ) + (\tan 68^\circ / \cot 22^\circ) + \cos^2 44^\circ + \cos^2 46^\circ$

$= (\sec 17^\circ / \operatorname{cosec}(90^\circ - 73^\circ)] + [\tan(90^\circ - 22^\circ) / \cot 22^\circ] + \cos^2(90^\circ - 44^\circ) + \cos^2 46^\circ$

$= [\sec 17^\circ / \sec 17^\circ] + [\cot 22^\circ / \cot 22^\circ] + [\sin^2 46^\circ + \cos^2 46^\circ]$

$= 1 + 1 + 1$

$= 3$

5.) i) $(\sin 65^\circ / \cos 25^\circ) + (\cos 32^\circ / \sin 58^\circ) - \sin 28^\circ \cdot \sec 62^\circ + \operatorname{cosec}^2 30^\circ$

→ Given that,

$(\sin 65^\circ / \cos 25^\circ) + (\cos 32^\circ / \sin 58^\circ) - \sin 28^\circ \cdot \sec 62^\circ + \operatorname{cosec}^2 30^\circ$

$= [\sin 65^\circ / \cos(90^\circ - 65^\circ)] + [\cos 32^\circ / \sin(90^\circ - 32^\circ)] - \sin 28^\circ \cdot \sec(90^\circ - 28^\circ) + 2^2$

$= [\sin 65^\circ / \sin 65^\circ] + [\cos 32^\circ / \cos 32^\circ] - (\sin 28^\circ \times \operatorname{cosec} 28^\circ] + 4$

$= 1 + 1 - 1 + 4$

$= 5$

ii) $(\sin 29^\circ / \operatorname{cosec} 61^\circ) + 2 \cot 8^\circ \cdot \cot 17^\circ \cdot \cot 45^\circ \cdot \cot 73^\circ \cdot \cot 82^\circ - 3(\sin^2 38^\circ + \sin^2 52^\circ)$.

→ Given that,

$(\sin 29^\circ / \operatorname{cosec} 61^\circ) + 2 \cot 8^\circ \cdot \cot 17^\circ \cdot \cot 45^\circ \cdot \cot 73^\circ \cdot \cot 82^\circ - 3(\sin^2 38^\circ + \sin^2 52^\circ)$

$= \sin 29^\circ / \operatorname{cosec}(90^\circ - 29^\circ) + [2 \cot 8^\circ \cdot \cot 17^\circ \cdot \cot 45^\circ \cdot \cot(90^\circ - 17^\circ) \cdot \cot(90^\circ - 8^\circ)] - 3[\sin^2 38^\circ + \sin^2(90^\circ - 38^\circ)]$

$= (\sin 29^\circ / \sin 29^\circ) + [2 \cot 8^\circ \cdot \cot 17^\circ \cdot \cot 45^\circ \cdot \tan 17^\circ \cdot \tan 8^\circ] - 3[\sin^2 38^\circ + \cos^2 38^\circ]$

$= 1 + 2[(\cot 8^\circ \cdot \tan 8^\circ)(\cot 17^\circ \cdot \tan 17^\circ) \cot 45^\circ] - 3$

$= 1 + 2(1 \times 1 \times 1) - 3$

$= 1 + 2 - 3$

$= 0$

6.) i) $(\sin 35^\circ \cdot \cos 55^\circ + \cos 35^\circ \cdot \sin 55^\circ) / (\operatorname{cosec}^2 10^\circ - \tan^2 80^\circ)$

→ Given that,

$[(\sin 35^\circ \cdot \cos 55^\circ + \cos 35^\circ \cdot \sin 55^\circ)] / (\operatorname{cosec}^2 10^\circ - \tan^2 80^\circ)$

$= [\sin 35^\circ \cdot \cos(90^\circ - 35^\circ) + \cos 35^\circ \cdot \sin(90^\circ - 35^\circ)] / [\operatorname{cosec}^2 10^\circ - \tan^2(90^\circ - 10^\circ)]$

$= [\sin 35^\circ \cdot \sin 35^\circ + \cos 35^\circ \cdot \cos 35^\circ] / [\operatorname{cosec}^2 10^\circ - \cot^2 10^\circ]$

$$= \frac{(\sin^2 35^\circ + \cos^2 35^\circ)}{(\operatorname{cosec}^2 10^\circ - \cot^2 10^\circ)} = \frac{(\sin^2 35^\circ + \cos^2 35^\circ)}{(\operatorname{cosec}^2 10^\circ - \cot^2 10^\circ)} = \frac{1}{1}$$

$= 1$

ii) $\sin^2 34^\circ + \sin^2 56^\circ + 2\tan 18^\circ \cdot \tan 72^\circ - \cot^2 30^\circ$

→ Given that,

$\sin^2 34^\circ + \sin^2 56^\circ + 2\tan 18^\circ \cdot \tan 72^\circ - \cot^2 30^\circ$

$= \sin^2 34^\circ + \sin^2(90^\circ - 34^\circ) + 2\tan 18^\circ \cdot \tan(90^\circ - 18^\circ) - \cot^2 30^\circ$

$= (\sin^2 34^\circ + \cos^2 34^\circ) + 2\tan 18^\circ \cdot \cot 18^\circ - \cot^2 30^\circ$

$= 1 + 2 \times 1 - (\sqrt{3})^2$

$= 1 + 2 - 3$

$= 0$

8.) Prove that: i) $(\sec A + \tan A)(1 - \sin A) = \cos A$

ii) $(1 + \tan^2 A)(1 - \sin A)(1 + \sin A) = 1$

→ i) L.H.S. $= (\sec A + \tan A)(1 - \sin A)$

$$= \left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)(1 - \sin A)$$

$$= \left(\frac{1 + \sin A}{\cos A}\right)(1 - \sin A)$$

$$= \frac{(1 + \sin A)(1 - \sin A)}{\cos A} = \frac{(1 - \sin^2 A)}{\cos A}$$

$$= \frac{\cos^2 A}{\cos A} \qquad \because \sin^2 A + \cos^2 A = 1$$

$= \cos A$

L.H.S. = R.H.S.

ii) $L.H.S. = (1+\tan^2 A)(1-\sin A)(1+\sin A)$

$= \left(1+\frac{\sin^2 A}{\cos^2 A}\right)(1-\sin^2 A)$ $\quad \because (a+b)(a-b) = a^2-b^2$

$= \frac{(\cos^2 A+\sin^2 A)}{\cos^2 A}(1-\sin^2 A)$

$= \frac{1}{\cos^2 A}(\cos^2 A)$

$\because \sin^2 A+\cos^2 A = 1$

$= 1$

$L.H.S. = R.H.S.$

9.) i) $\tan A + \cot A = \sec A \cdot \operatorname{cosec} A$

ii) $(1-\cos A)(1+\sec A) = \tan A \cdot \sin A$

→ i) $L.H.S. = \tan A + \cot A$ $\quad \because \frac{1}{\sin A} = \operatorname{cosec} A$

$= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$ $\quad \because \frac{1}{\cos A} = \sec A$

$= \frac{(\sin^2 A+\cos^2 A)}{\sin A \cdot \cos A} = \frac{1}{\sin A \cdot \cos A}$ $\quad \because \sin^2 A+\cos^2 A = 1$

$= \frac{1}{\sin A} \cdot \frac{1}{\cos A} = \operatorname{cosec} A \cdot \sec A$

$= \sec A \cdot \operatorname{cosec} A$

$L.H.S. = R.H.S.$

ii) $(1-\cos A)(1+\sec A) = (1-\cos A)\left(1+\frac{1}{\cos A}\right)$ $\quad \because \sec A = \frac{1}{\cos A}$

$= (1-\cos A)\frac{(\cos A+1)}{\cos A}$

$= \frac{(1-\cos A)(1+\cos A)}{\cos A}$

$= (1-\cos^2 A) \cdot \frac{1}{\cos A}$ $\quad \because (a-b)(a+b) = a^2-b^2$

$= \sin^2 A \cdot \frac{1}{\cos A}$ $\quad \because \sin^2 A+\cos^2 A = 1$

$= \frac{\sin A}{\cos A} \cdot \sin A$

$(1-\cos A)(1+\sec A) = \tan A \cdot \sin A$

10.) i) $\cot^2A - \cos^2A = \cot^2A \cdot \cos^2A$

ii) $1 + \frac{\tan^2A}{(1+\sec A)} = \sec A$

iii) $\frac{(1+\sec A)}{\sec A} = \frac{\sin^2A}{(1-\cos A)}$

iv) $\frac{\sin A}{(1-\cos A)} = \operatorname{cosec}A + \cot A$

$\rightarrow$ i) $\cot^2A - \cos^2A = \frac{\cos^2A}{\sin^2A} - \cos^2A$ $\quad \because \cot A = \frac{\cos A}{\sin A}$

$$= \left(\frac{1}{\sin^2A} - 1\right)\cos^2A$$

$$= \frac{(1-\sin^2A)}{\sin^2A} \cdot \cos^2A$$

$$= \frac{\cos^2A}{\sin^2A} \cdot \cos^2A \quad \because \sin^2A + \cos^2A = 1$$

$\boxed{\cot^2A - \cos^2A = \cot^2A \cdot \cos^2A}$ Hence proved.

ii) $1 + \frac{\tan^2A}{(1+\sec A)} = \sec A$

$1 + \frac{\tan^2A}{1+\sec A} = 1 + \frac{(\sec^2A - 1)}{(1+\sec A)}$ $\quad \because 1 + \tan^2A = \sec^2A$, $\tan^2A = \sec^2A - 1$

$$= \frac{1 + \sec A + \sec^2A - 1}{1 + \sec A}$$

$$= \frac{\sec A(1+\sec A)}{(1+\sec A)}$$

$\boxed{1 + \frac{\tan^2A}{(1+\sec A)} = \sec A}$ Hence proved.

iii) $\frac{(1+\sec A)}{\sec A} = \frac{1}{\sec A} + 1$

$$= \cos A + 1$$

$$= \frac{(1+\cos A)(1-\cos A)}{(1-\cos A)}$$

$$= \frac{1-\cos^2A}{(1-\cos A)} = \frac{\sin^2A}{(1-\cos A)}$$

$\boxed{\frac{(1+\sec A)}{\sec A} = \frac{\sin^2A}{(1-\cos A)}}$ Hence proved.

iv) $\frac{\sin A}{(1-\cos A)} = \frac{\sin A\,(1+\cos A)}{(1-\cos A)\,(1+\cos A)}$ multiply D^{er} & N^{er} by $(1+\cos A)$

$$= \frac{\sin A\,(1+\cos A)}{(1-\cos^2 A)}$$

$$= \frac{\sin A\,(1+\cos A)}{\sin^2 A} \qquad \because \cos^2 A + \sin^2 A = 1$$

$$= \frac{1}{\sin A}\,(1+\cos A)$$

$$= \frac{1}{\sin A} + \frac{\cos A}{\sin A}$$

$$\boxed{\frac{\sin A}{(1-\cos A)} = \operatorname{cosec} A + \cot A}$$ Hence proved.

II.) i) $\frac{\sin A}{(1+\cos A)} = \frac{(1-\cos A)}{\sin A}$

$\rightarrow$ $\frac{\sin A}{(1+\cos A)} = \frac{\sin A\,(1-\cos A)}{(1+\cos A)\,(1-\cos A)}$ $\because$ multiply D^{er} & N^{er} by $(1-\cos A)$

$$= \frac{\sin A\,(1-\cos A)}{(1-\cos^2 A)} \qquad \because (a-b)(a+b) = a^2-b^2$$

$$= \frac{\sin A\cdot(1-\cos A)}{\sin^2 A} \qquad \because \sin^2 A + \cos^2 A = 1$$

$$\boxed{\frac{\sin A}{(1+\cos A)} = \frac{(1-\cos A)}{\sin A}}$$ Hence proved

ii) $\frac{(1-\tan^2 A)}{(\cot^2 A - 1)} = \tan^2 A$

$\rightarrow$ $\frac{(1-\tan^2 A)}{(\cot^2 A - 1)} = \frac{1 - \sin^2 A/\cos^2 A}{\frac{\cos^2 A}{\sin^2 A} - 1}$

$$= \frac{(\cos^2 A - \sin^2 A)/\cos^2 A}{(\cos^2 A - \sin^2 A)/\sin^2 A}$$

$$= \frac{1/\cos^2 A}{1/\sin^2 A} = \frac{\sin^2 A}{\cos^2 A} = \tan^2 A$$ Hence proved

iii) $\frac{\sin A}{(1+\cos A)} = \operatorname{cosec} A - \cot A$

$\rightarrow$ $\frac{\sin A}{(1+\cos A)} = \frac{\sin A(1-\cos A)}{(1+\cos A)(1-\cos A)}$

$= \frac{\sin A(1-\cos A)}{(1-\cos^2 A)}$

$= \frac{\sin A(1-\cos A)}{\sin^2 A}$ $\quad \because \sin^2 A+\cos^2 A = 1$

$= \frac{(1-\cos A)}{\sin A} = \frac{1}{\sin A} - \frac{\cos A}{\sin A}$

$\boxed{\frac{\sin A}{(1+\cos A)} = \operatorname{cosec} A - \cot A}$ Hence proved.

12) i) $\frac{(\sec A-1)}{(\sec A+1)} = \frac{(1-\cos A)}{(1+\cos A)}$

$\rightarrow$ $\frac{(\sec A-1)}{(\sec A+1)} = \frac{1/\cos A - 1}{\frac{1}{\cos A}+1}$

$= \frac{(1-\cos A)/\cos A}{(1+\cos A)/\cos A}$

$\boxed{\frac{(\sec A-1)}{(\sec A+1)} = \frac{(1-\cos A)}{(1+\cos A)}}$ Hence proved.

ii) $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \cdot \operatorname{cosec}^2 A$

$\rightarrow$ $\sec^2 A + \operatorname{cosec}^2 A = \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}$

$= \frac{(\sin^2 A + \cos^2 A)}{\sin^2 A \cdot \cos^2 A}$

$= \frac{1}{\sin^2 A \cdot \cos^2 A}$

$= \frac{1}{\sin^2 A} \cdot \frac{1}{\cos^2 A}$

$= \operatorname{cosec}^2 A \cdot \sec^2 A$

$\boxed{(\sec^2 A + \operatorname{cosec}^2 A) = \sec^2 A \cdot \operatorname{cosec}^2 A}$ Hence proved.

18) i) $\frac{(1+\sin A)}{\cos A} + \frac{\cos A}{(1+\sin A)} = 2\sec A$

$\longrightarrow$

$$\frac{(1+\sin A)}{\cos A} + \frac{\cos A}{(1+\sin A)} = \frac{(1+\sin A)^2 + \cos^2 A}{\cos A(1+\sin A)}$$

$$= \frac{1+2\sin A+\sin^2 A+\cos^2 A}{\cos A + \cos A \cdot \sin A}$$

$$= \frac{1+2\sin A+1}{\cos A(1+\sin A)}$$

$$= \frac{2+2\sin A}{\cos A(1+\sin A)}$$

$$= \frac{2(1+\sin A)}{\cos A \cdot (1+\sin A)}$$

$$= \frac{2}{\cos A}$$

$$\boxed{\frac{(1+\sin A)}{\cos A} + \frac{\cos A}{(1+\sin A)} = 2\sec A}$$

Hence proved.

ii) $\frac{\tan A}{(\sec A-1)} + \frac{\tan A}{(\sec A+1)} = 2\operatorname{cosec} A$

$\longrightarrow$

$$\frac{\tan A}{(\sec A-1)} + \frac{\tan A}{(\sec A+1)} = \tan A\left[\frac{1}{\sec A-1} + \frac{1}{\sec A+1}\right]$$

$$= \tan A\left[\frac{\sec A+1+\sec A-1}{(\sec A+1)(\sec A-1)}\right]$$

$$= \tan A\left[\frac{2\sec A}{(\sec^2 A-1)}\right]$$

$\because 1+\tan^2 A = \sec^2 A$

$$= \frac{2\tan A \cdot \sec A}{\tan^2 A}$$

$$= \frac{2\sec A}{\tan A}$$

$$= \frac{2/\cos A}{\sin A/\cos A}$$

$$= \frac{2}{\sin A}$$

$$\boxed{\frac{\tan A}{(\sec A-1)} + \frac{\tan A}{(\sec A+1)} = 2\operatorname{cosec} A}$$

Hence proved.

14) i) $\cot A - \tan A = (2\cos^2 A - 1)/(\sin A \cdot \cos A)$

$\longrightarrow \cot A - \tan A = \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A}$

$= \frac{\cos^2 A - \sin^2 A}{\sin A \cdot \cos A}$

$= \frac{\cos^2 A - (1 - \cos^2 A)}{\sin A \cdot \cos A}$ $\quad \because \sin^2 A + \cos^2 A = 1$

$= \frac{\cos^2 A - 1 + \cos^2 A}{\sin A \cdot \cos A}$

$\boxed{\cot A - \tan A = \frac{(2\cos^2 A - 1)}{\sin A \cdot \cos A}}$ Hence proved.

ii) $\frac{\cot A - 1}{2 - \sec^2 A} = \frac{\cot A}{1 + \tan A}$

$\longrightarrow \frac{\cot A - 1}{2 - \sec^2 A} = \frac{\frac{\cos A}{\sin A} - 1}{2 - \frac{1}{\cos^2 A}}$

$= \frac{(\cos A - \sin A)/\sin A}{(2\cos^2 A - 1)/\cos^2 A} = \frac{\cos^2 A (\cos A - \sin A)}{\sin A (2\cos^2 A - 1)}$

$= \frac{\cos^2 A (\cos A - \sin A)}{\sin A [2\cos^2 A - \sin^2 A - \cos^2 A]}$

$= \frac{\cos^2 A (\cos A - \sin A)}{\sin A [\cos^2 A - \sin^2 A)} = \frac{\cos^2 A (\cos A - \sin A)}{\sin A (\cos A - \sin A)(\cos A + \sin A)}$

$= \frac{\cos^2 A}{\sin A (\cos A + \sin A)}$

$= \frac{\cos A / \sin A}{(\cos A + \sin A)/\cos A}$

$= \frac{\cot A}{(1 + \sin A/\cos A)}$

$\boxed{\frac{\cot A - 1}{2 - \sec^2 A} = \frac{\cot A}{1 + \tan A}}$ Hence proved.

15) i) $\tan^2\theta - \sin^2\theta = \tan^2\theta \cdot \sin^2\theta$

$\longrightarrow$ $\tan^2\theta - \sin^2\theta = \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta$

$= \frac{\sin^2\theta - \sin^2\theta(\cos^2\theta)}{\cos^2\theta}$

$= \frac{\sin^2\theta\ (1-\cos^2\theta)}{\cos^2\theta}$

$= \frac{\sin^2\theta}{\cos^2\theta}(1-\cos^2\theta)$

$\boxed{\tan^2\theta - \sin^2\theta = \tan^2\theta \cdot \sin^2\theta}$ $\quad \because \sin^2\theta + \cos^2\theta = 1$

Hence proved.

ii) $\frac{\cos\theta}{(1-\tan\theta)} - \frac{\sin^2\theta}{(\cos\theta - \sin\theta)} = \cos\theta + \sin\theta$

$\longrightarrow$ $\frac{\cos\theta}{(1-\tan\theta)} - \frac{\sin^2\theta}{(\cos\theta-\sin\theta)} = \frac{\cos\theta}{\left(1-\frac{\sin\theta}{\cos\theta}\right)} - \frac{\sin^2\theta}{(\cos\theta-\sin\theta)}$

$= \frac{\cos\theta}{\frac{(\cos\theta-\sin\theta)}{\cos\theta}} - \frac{\sin^2\theta}{(\cos\theta-\sin\theta)}$

$= \frac{\cos^2\theta}{(\cos\theta-\sin\theta)} - \frac{\sin^2\theta}{(\cos\theta-\sin\theta)}$

$= \frac{(\cos^2\theta-\sin^2\theta)}{(\cos\theta-\sin\theta)}$

$= \frac{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}{(\cos\theta-\sin\theta)}$

$\boxed{\frac{\cos\theta}{(1-\tan\theta)} - \frac{\sin^2\theta}{(\cos\theta-\sin\theta)} = \cos\theta+\sin\theta}$ $\quad \because (a-b)(a+b) = a^2-b^2$

Hence proved.

16) i) $\operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta = \cot^4\theta + \cot^2\theta$

$\longrightarrow$ $\operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta = \operatorname{cosec}^2\theta\,(\operatorname{cosec}^2\theta - 1)$

$= \operatorname{cosec}^2\theta \cdot \cot^2\theta$ $\quad \because 1+\cot^2\theta = \operatorname{cosec}^2\theta$

$= (1+\cot^2\theta)\cdot\cot^2\theta$

$= \cot^2\theta + \cot^4\theta$

$\boxed{\operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta = \cot^4\theta + \cot^2\theta}$ Hence proved.

ii) $2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta = \cot^4\theta - \tan^4\theta$

$\longrightarrow$

$= 2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta$

$= \cancel{\sec^2\theta}\ 2(1+\tan^2\theta) - (1+\tan^2\theta)^2 - 2(1+\cot^2\theta) + (1+\cot^2\theta)^2$

$= 2\tan^2\theta + 2 - (\tan^4\theta + 2\tan^2\theta + 1) - 2\cot^2\theta - 2 + (\cot^4\theta + 2\cot^2\theta + 1)$

$= \cot^4\theta - \tan^4\theta$

$=$ R.H.S.

17) i) $\dfrac{(1+\cos\theta - \sin^2\theta)}{\sin\theta\,(1+\cos\theta)} = \cot\theta$

$\longrightarrow$ $\dfrac{1+\cos\theta - \sin^2\theta}{\sin\theta\,(1+\cos\theta)} = \dfrac{(1-\sin^2\theta)+\cos\theta}{\sin\theta\,(1+\cos\theta)}$

$= \dfrac{\cos^2\theta + \cos\theta}{(1+\cos\theta)\sin\theta}$ $\quad \because \sin^2\theta + \cos^2\theta = 1$

$= \dfrac{\cos\theta}{\sin\theta}\,\dfrac{(1+\cos\theta)}{(1+\cos\theta)}$

$\boxed{\dfrac{1+\cos\theta - \sin^2\theta}{\sin\theta\,(1+\cos\theta)} = \cot\theta}$ Hence proved.

ii) $\dfrac{(\tan^3\theta - 1)}{(\tan\theta - 1)} = \sec^2\theta + \tan\theta$

$\rightarrow$ $\dfrac{\tan^3\theta - 1}{\tan\theta - 1} = \dfrac{(\tan^3\theta - 1^3)}{(\tan\theta - 1)}$ $\quad \because (a^3 - b^3) = (a-b)(a^2 + ab + b^2)$

$= \dfrac{(\tan\theta - 1)}{(\tan\theta - 1)}(\tan^2\theta + \tan\theta + 1)$

$= (1 + \tan^2\theta + \tan\theta)$ $\quad \because 1 + \tan^2\theta = \sec^2\theta$

$\boxed{\dfrac{\tan^3\theta - 1}{\tan\theta - 1} = (\sec^2\theta + \tan\theta)}$ Hence proved.

18) i) $\dfrac{(1 + \operatorname{cosec} A)}{\operatorname{cosec} A} = \dfrac{\cos^2 A}{(1 - \sin A)}$

$\rightarrow$

$\dfrac{1 + \operatorname{cosec} A}{\operatorname{cosec} A} = \dfrac{1 + 1/\sin A}{1/\sin A}$

$= \dfrac{(\sin A + 1)/\sin A}{1/\sin A}$

$= \dfrac{(1 + \sin A)(1 - \sin A)}{(1 - \sin A)}$

$= \dfrac{(1 - \sin^2 A)}{(1 - \sin A)} = \dfrac{\cos^2 A}{(1 - \sin A)}$ $\quad \because \cos^2 A + \sin^2 A = 1$

$\boxed{\dfrac{(1 + \operatorname{cosec} A)}{\operatorname{cosec} A} = \dfrac{\cos^2 A}{(1 - \sin A)}}$ Hence proved.

ii) $\sqrt{\dfrac{(1 - \cos A)}{(1 + \cos A)}} = \operatorname{cosec} A - \cot A$

$\rightarrow$ $\sqrt{\dfrac{(1 - \cos A)}{(1 + \cos A)}} = \sqrt{\dfrac{(1 - \cos A)(1 + \cos A)}{(1 + \cos A)(1 - \cos A)}}$

$= \sqrt{\dfrac{(1 - \cos A)^2}{(1 - \cos^2 A)}}$

$= \sqrt{\dfrac{(1 - \cos A)^2}{\sin^2 A}} = \dfrac{1 - \cos A}{\sin A} = \dfrac{1}{\sin A} - \dfrac{\cos A}{\sin A}$

$= \operatorname{cosec} A - \cot A$. Hence proved

19.) i) $\sqrt{\frac{1+\sin A}{1-\sin A}} = \tan A + \sec A$

$\longrightarrow$

$$\sqrt{\frac{(1+\sin A)}{(1-\sin A)}} = \sqrt{\frac{(1+\sin A)(1+\sin A)}{(1-\sin A)(1+\sin A)}}$$

$$= \sqrt{\frac{(1+\sin A)^2}{(1-\sin^2 A)}}$$

$$= \sqrt{\frac{(1+\sin A)^2}{\cos^2 A}} = \sqrt{\frac{(1+\sin A)^2}{\cos^2 A}} = \frac{(1+\sin A)}{\cos A}$$

$$= \frac{1}{\cos A} + \frac{\sin A}{\cos A}$$

$$= \sec A + \tan A$$

$$\boxed{\sqrt{\frac{(1+\sin A)}{(1-\sin A)}} = \tan A + \sec A}$$

Hence proved,

ii) $\sqrt{\frac{(1-\cos A)}{(1+\cos A)}} = \operatorname{cosec} A - \cot A$

$\longrightarrow$

$$\sqrt{\frac{(1-\cos A)}{(1+\cos A)}} = \sqrt{\frac{(1-\cos A)(1-\cos A)}{(1+\cos A)(1-\cos A)}}$$

$$= \sqrt{\frac{(1-\cos A)^2}{(1-\cos^2 A)}}$$

$$= \sqrt{\frac{(1-\cos A)^2}{\sin^2 A}} = \frac{(1-\cos A)}{\sin A}$$

$$= \frac{1}{\sin A} - \frac{\cos A}{\sin A}$$

$$\boxed{\sqrt{\frac{(1-\cos A)}{(1+\cos A)}} = \operatorname{cosec} A - \cot A}$$

Hence proved.

20) i) $\sqrt{\frac{\sec A-1}{\sec A+1}}+\sqrt{\frac{\sec A+1}{\sec A-1}}=2\operatorname{cosec}A$

$\rightarrow$

$$\sqrt{\frac{\sec A-1}{\sec A+1}}+\sqrt{\frac{\sec A+1}{\sec A-1}}=\sqrt{\frac{(\sec A-1)(\sec A-1)}{(\sec A+1)(\sec A-1)}}+\sqrt{\frac{(\sec A+1)(\sec A+1)}{(\sec A-1)(\sec A+1)}}$$

$$=\sqrt{\frac{(\sec A-1)^2}{(\sec^2A-1)}}+\sqrt{\frac{(\sec A+1)^2}{(\sec^2A-1)}}$$

$$=\sqrt{\frac{(\sec A-1)^2}{\tan^2A}}+\sqrt{\frac{(\sec A+1)^2}{\tan^2A}} \quad \because 1+\tan^2A=\sec^2A$$

$$=\frac{(\sec A-1)}{\tan A}+\frac{(\sec A+1)}{\tan A}=\frac{\sec A-1+\sec A+1}{\tan A}$$

$$=\frac{2\sec A}{\tan A}=\frac{2/\cos A}{\sin A/\cos A}$$

$$=\frac{2}{\sin A}=2\operatorname{cosec}A$$

L.H.S. = R.H.S. Hence proved

ii) $\frac{\cos A\cdot\cot A}{(1-\sin A)}=1+\operatorname{cosec}A$

$\rightarrow$

$$\frac{\cos A\cdot\cot A}{(1-\sin A)}=\frac{\cos A\cdot\cos A/\sin A}{(1-\sin A)}$$

$$=\frac{\cos^2A/\sin A}{(1-\sin A)}=\frac{\cos^2A}{\sin A\cdot(1-\sin A)}$$

$$=\frac{1-\sin^2A}{\sin A\cdot(1-\sin A)}=\frac{(1-\sin A)(1+\sin A)}{\sin A\cdot(1-\sin A)}$$

$$=\frac{(1+\sin A)}{\sin A}$$

$$=\frac{1}{\sin A}+1$$

$$=\operatorname{cosec}A+1$$

$$\boxed{\frac{\cos A\cdot\cot A}{(1-\sin A)}=\operatorname{cosec}A+1}$$ Hence proved.

23) $\dfrac{\cot\theta + \csc\theta - 1}{(\cot\theta - \csc\theta + 1)} = \dfrac{(1+\cos\theta)}{\sin\theta}$

$\rightarrow \dfrac{(\cot\theta - \csc\theta - 1)}{(\cot\theta - \csc\theta + 1)} = \dfrac{\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} - 1}{\frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta} + 1}$

$= \dfrac{\cos\theta + 1 - \sin\theta}{\cos\theta - 1 + \sin\theta}$

$= \dfrac{\cos\theta + (1+\sin\theta)}{\cos\theta - (1-\sin\theta)}$

$= \dfrac{[\cos\theta + (1-\sin\theta)]}{[\cos\theta - (1-\sin\theta)]} \dfrac{[\cos\theta + (1-\sin\theta)]}{[\cos\theta + (1-\sin\theta)]}$

$= \dfrac{[\cos\theta + (1-\sin\theta)]^2}{\cos^2\theta - (1-\sin\theta)^2}$

$= \dfrac{\cos^2\theta + (1-\sin\theta)^2 + 2\cos\theta(1-\sin\theta)}{\cos^2\theta - (1 - 2\sin\theta + \sin^2\theta)}$

$= \dfrac{\cos^2\theta + 1 - 2\sin\theta + \sin^2\theta + 2\cos\theta - 2\sin\theta\cdot\cos\theta}{\cos^2\theta - 1 + 2\sin\theta - \sin^2\theta}$

$= \dfrac{2 - 2\sin\theta + 2\cos\theta - 2\sin\theta\cdot\cos\theta}{1 - \sin^2\theta - 1 + 2\sin\theta - \sin^2\theta}$

$= \dfrac{2 + 2\cos\theta - 2\sin\theta - 2\sin\theta\cos\theta}{2\sin\theta - 2\sin^2\theta}$

$= \dfrac{2(1+\cos\theta) - 2\sin\theta(1+\cos\theta)}{2\sin\theta(1-\sin\theta)}$

$= \dfrac{(1+\cos\theta)(2-2\sin\theta)}{2\sin\theta\cdot(1-\sin\theta)} = \dfrac{2(1+\cos\theta)(1-\sin\theta)}{2\sin\theta(1-\sin\theta)}$

$= \dfrac{1+\cos\theta}{\sin\theta}$

L.H.S. = R.H.S.

Hence proved.

24.) $(\sin\theta+\cos\theta)(\sec\theta+\csc\theta)=2+\sec\theta\cdot\csc\theta$

$$\rightarrow (\sin\theta+\cos\theta)(\sec\theta+\csc\theta)=(\sin\theta+\cos\theta)\left(\frac{1}{\cos\theta}+\frac{1}{\sin\theta}\right)$$

$$=(\sin\theta+\cos\theta)\frac{(\sin\theta+\cos\theta)}{\sin\theta\cdot\cos\theta}$$

$$=\frac{(\sin\theta+\cos\theta)^2}{\sin\theta\cdot\cos\theta}$$

$$=\frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cdot\cos\theta}{\sin\theta\cdot\cos\theta}$$

$$=\frac{1+2\sin\theta\cdot\cos\theta}{\sin\theta\cdot\cos\theta}$$

$$=\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}+2$$

$$=\csc\theta\cdot\sec\theta+2$$

$$\boxed{(\sin\theta+\cos\theta)(\sec\theta+\csc\theta)=2+\sec\theta\cdot\csc\theta}$$

Hence proved.

25.) $\frac{\tan^2A}{1+\tan^2A}+\frac{\cot^2A}{1+\cot^2A}=1$

$$\rightarrow \frac{\tan^2A}{(1+\tan^2A)}+\frac{\cot^2A}{(1+\cot^2A)}=\frac{\tan^2A}{(1+\tan^2A)}+\frac{1/\tan^2A}{1+1/\tan^2A}$$

$$=\frac{\tan^2A}{(1+\tan^2A)}+\frac{1/\tan^2A}{(\tan^2A+1)/\tan^2A}$$

$$=\frac{(\tan^2A)}{(1+\tan^2A)}+\frac{1}{(1+\tan^2A)}$$

$$=\frac{(1+\tan^2A)}{(1+\tan^2A)}$$

$$=1$$

L.H.S. = R.H.S.

Hence proved.

26) i) $\frac{1}{\sec A + \tan A} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{(\sec A - \tan A)}$

$\longrightarrow \frac{1}{(\sec A + \tan A)} - \frac{1}{\cos A} = \frac{1}{\frac{1}{\cos A} + \frac{\sin A}{\cos A}} - \frac{1}{\cos A}$

$= \frac{\cos A}{(1+\sin A)} - \frac{1}{\cos A}$

$= \frac{\cos^2 A - (1+\sin A)}{\cos A\,(1+\sin A)} =$

$= - \frac{1}{\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)} - \frac{1}{\cos A}$

$= \frac{-1}{\cos A} + \frac{1}{(\sec A + \tan A)}$

L.H.S. = R.H.S. Hence proved

ii) $\frac{(\tan A + \sin A)}{(\tan A - \sin A)} = \frac{(\sec A + 1)}{(\sec A - 1)}$

$\longrightarrow \frac{(\tan A + \sin A)}{(\tan A - \sin A)} = \frac{(\sin A/\cos A + \sin A)}{\left(\frac{\sin A}{\cos A} - \sin A\right)}$

$= \frac{\sin A\,(1/\cos A + 1)}{\sin A\,(1/\cos A - 1)}$

$= \frac{(\sec A + 1)}{(\sec A - 1)}$

L.H.S. = R.H.S. Hence proved.

27) If $(\sin\theta + \cos\theta) = \sqrt{2}\sin(90-\theta)$, show that $\cot\theta = \sqrt{2}+1$

$\rightarrow$ Given that, $\sin\theta + \cos\theta = \sqrt{2}\cdot\sin(90-\theta)$

$$\sin\theta + \cos\theta = \sqrt{2}\cos\theta$$

$$\Rightarrow \frac{(\sin\theta+\cos\theta)}{\sin\theta} = \sqrt{2}\frac{\cos\theta}{\sin\theta}$$

$$1+\cot\theta = \sqrt{2}\cot\theta$$

$$1 = (\sqrt{2}-1)\cot\theta$$

$$\cot\theta = \frac{1}{(\sqrt{2}-1)} = \frac{(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{(2-1)}$$

$\boxed{\cot\theta = \sqrt{2}+1}$ Hence proved.

28) If $7(\sin^2\theta) + 3\cos^2\theta = 4$, $0^\circ \le \theta \le 90^\circ$, then find the value of θ.

$\rightarrow$ Given that, $7\sin^2\theta + 3\cos^2\theta = 4$, $0 \le \theta \le 90$

$$3\sin^2\theta + 4\sin^2\theta + 3\cos^2\theta = 4$$

$$(3\sin^2\theta + 3\cos^2\theta) + 4\sin^2\theta = 4$$

$$3(\sin^2\theta + \cos^2\theta) + 4\sin^2\theta = 4$$

$$3 + 4\sin^2\theta = 4$$

$$4\sin^2\theta = 1$$

$$\sin^2\theta = 1/4$$

$$\sin\theta = 1/2$$

$$\theta = \sin^{-1}(1/2)$$

$\boxed{\theta = 30^\circ}$ is the required value of θ.

29.) If $(\sec\theta + \tan\theta) = m$ and $(\sec\theta - \tan\theta) = n$, prove that $m \times n = 1$.

$\rightarrow$ Given that, $(\sec\theta + \tan\theta) = m$ and $(\sec\theta - \tan\theta) = n$

$$\begin{aligned} L.H.S. &= m \times n \\ &= (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) \\ &= \sec^2\theta - \tan^2\theta \qquad \because 1 + \tan^2\theta = \sec^2\theta \\ &= 1 \\ &= R.H.S. \end{aligned}$$

$\Rightarrow$ $\boxed{mn = 1}$ Hence proved.

30.) If $x = a\sec\theta + b\tan\theta$ and $y = a\tan\theta + b\sec\theta$, prove that $x^2 - y^2 = a^2 - b^2$.

$\rightarrow$ Given that, $x = a\sec\theta + b\tan\theta$ & $y = a\tan\theta + b\sec\theta$

$$\begin{aligned} \text{Then, } L.H.S. &= x^2 - y^2 \\ &= (a\sec\theta + b\tan\theta)^2 - (a\tan\theta + b\sec\theta)^2 \\ &= a^2\sec^2\theta + b^2\tan^2\theta + 2ab\sec\theta.\tan\theta - \\ &\quad (a^2\tan^2\theta + b^2\sec^2\theta + 2ab\tan\theta.\sec\theta) \\ &= a^2\sec^2\theta + b^2\tan^2\theta + 2ab\sec\theta.\tan\theta - a^2\tan^2\theta \\ &\quad - b^2\sec^2\theta - 2ab\sec\theta.\tan\theta \\ &= a^2(\sec^2\theta - \tan^2\theta) + b^2(\tan^2\theta - \sec^2\theta) \\ &= a^2(1) + b^2(-1) \qquad \because 1 + \tan^2\theta = \sec^2\theta \\ &= a^2 - b^2 \\ &= R.H.S. \end{aligned}$$

Thus, $\boxed{x^2 - y^2 = a^2 - b^2}$ Hence proved.